

Thresholds for correcting errors, erasures, and faulty syndrome measurements in degenerate quantum codes

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We suggest a technique for constructing lower (existence) bounds for the fault-tolerant threshold to scalable quantum computation applicable to degenerate quantum codes with sublinear distance scaling. We give explicit analytic expressions combining probabilities of erasures, depolarizing errors, and phenomenological syndrome measurement errors for quantum LDPC codes with logarithmic or larger distances. These threshold estimates are parametrically better than the existing analytical bound based on percolation.

Quantum computers are (in theory) faster than the classical ones because of the quantum parallelism. Instead of working with sets of classical bits which can store one binary string at a time, a quantum computer operates with coherent superpositions of exponentially many basis states encoded in qubits. Such superpositions are extremely fragile: decoherence due to environment or intrinsic errors would make quantum computation unfeasible, were it not for quantum error correction[1]. An important result is the *threshold theorem* stating that with physical qubits and elementary gates exceeding some accuracy threshold, arbitrarily large quantum computation is possible with at most polynomial hardware cost[2–10].

In any quantum error-correcting code (QECC), certain measurements have to be done repeatedly. Unlike classical communications setup where errors happen only during the transmission, this *syndrome* extraction from a system of qubits is a complicated measurement prone to errors. It requires fault-tolerance (FT): all operations have to be specially designed to limit error propagation. Requirement of FT severely restricts the codes which can be used in a quantum computer. Even though many families of QECCs have been constructed[11, 12], for many years, FT was demonstrated for only two code families, concatenated[2] and surface[5] codes (as well as related color codes[13–15]). Both families require substantial hardware overhead, in technical terms, they have asymptotically zero rates[16].

So far the only family of QECCs where *finite* rates and an FT threshold to scalable quantum computation are known to coexist are the quantum LDPC codes[17, 18]. These are just stabilizer codes[11, 19] where stabilizer generators (operators to be measured during QEC) involve a limited number of qubits each. Several finite-rate families of such codes are known[20–24]. The threshold existence has been proved[25] by two of us using ideas from percolation theory. Subsequently, a related approach has been used by Gottesman[26] who demonstrated that with such codes, scalable quantum computation is possible with a finite overhead per logical qubit.

While the technique in Ref. 25 shows the existence of

a finite threshold for certain quantum LDPC codes, the actual threshold value and its dependence on the parameters are both far off. The technique[25] is also too restrictive: it fails to give a finite threshold whenever a single qubit is shared by many stabilizer generators.

In this work we present an approach resulting in a parametrically more accurate lower bound for the threshold, both in the setting of a quantum channel and in the FT setting using a phenomenological error model. We consider quantum LDPC codes whose distances scale as or faster than a logarithm of the code length n , while all stabilizer generators are limited to some fixed number w or fewer qubits. For any sequence of such codes, we give an analytical lower (existence) bound combining uncorrelated qubit erasures, depolarizing errors, and syndrome measurement errors. We also give a similar bound tailored for CSS codes. These bounds no longer require that every qubit be included in a limited number of stabilizer generators. Tying our lower bound on erasure threshold with other results[27, 28], we restrict the parameter space for codes with certain properties. This approach could also help analyzing FT for other degenerate code families, e.g., constructed recently by Bravyi and Hastings[24].

We consider QECCs defined on the n -qubit Hilbert space $\mathcal{H}_2^{\otimes n}$, where \mathcal{H}_2 is the single-qubit complex Hilbert space spanned by two orthonormal states $\{|0\rangle, |1\rangle\}$. Any operator acting in $\mathcal{H}_2^{\otimes n}$ can be represented as a linear combination of *Pauli operators*, elements of the n -qubit Pauli group \mathcal{P}_n of size 2^{2n+2} ,

$$\mathcal{P}_n = i^m \{I, X, Y, Z\}^{\otimes n}, \quad m = 0, \dots, 3, \quad (1)$$

where X , Y , and Z are the usual Pauli matrices, I is the identity matrix, and i^m a phase factor. *Weight* $\text{wt } E$ of a Pauli operator $E \in \mathcal{P}_n$ is the number of non-identity terms in its expansion (1). A *stabilizer code* \mathcal{Q} with parameters $[[n, k, d]]$ is a 2^k -dimensional subspace of the Hilbert space $\mathcal{H}_2^{\otimes n}$. \mathcal{Q} is a common $+1$ eigenspace of operators in an Abelian *stabilizer group* $\mathcal{S} = \langle G_1, \dots, G_r \rangle$, $-1 \notin \mathcal{S}$, with $r \equiv n - k$ generators G_i ,

$$\mathcal{Q} = \{|\psi\rangle : S|\psi\rangle = |\psi\rangle, \forall S \in \mathcal{S}\}. \quad (2)$$

A more narrow set of Calderbank-Shor-Steane (CSS) codes [29, 30] contains codes whose stabilizer generators can be chosen as products of only Pauli X or Pauli Z operators each. For a stabilizer group with r independent generators, the dimension of the quantum code is given by $k = n - r$; for a CSS code with r_X generators of X -type and r_Z generators of Z type we have $k = n - r_X - r_Z$.

The error correction is done by measuring the stabilizer generators $G_i, i = 1, \dots, r$; the corresponding eigenvalues $(-1)^{s_i}, s_i \in \{0, 1\}$ form the *syndrome* $\mathbf{s} \equiv (s_1, s_2, \dots, s_r)$ of the error. Measuring the syndrome projects any state $|\psi\rangle \in \mathcal{H}_2^{\otimes n}$ into one of the 2^r subspaces $\mathcal{Q}_\mathbf{s}$ equivalent to the code $\mathcal{Q} \equiv \mathcal{Q}_0$. An error $E \in \mathcal{P}_n$ is called detectable if it anticommutes with any generator of the stabilizer; otherwise it is called undetectable. Then, for any $|\psi\rangle \in \mathcal{Q}$, the syndrome measured in the state $E|\psi\rangle$ is non-zero for a detectable error and it is zero otherwise. While operators in the stabilizer group are undetectable, they act trivially on the code; such errors can be ignored. Any two Pauli errors E_1, E_2 which differ by a phase and an element of the stabilizer, $E_2 = e^{i\alpha} E_1 S, S \in \mathcal{S}$, are called degenerate. Mutually degenerate errors act identically on the code, they cannot (and need not) be distinguished.

The distance d of the code \mathcal{Q} is given by the minimum weight of an undetectable Pauli error $E \in \mathcal{P}_n$ which is not a part of the stabilizer, $E \notin \mathcal{S}$. A code with distance d can detect any Pauli error of weight up to $d - 1$, and it can correct any Pauli error of weight up to $\lfloor d/2 \rfloor$.

A code is called *degenerate* if its stabilizer includes a non-trivial operator $S \in \mathcal{S}$ with weight smaller than the distance, $0 \neq \text{wgt } S < d$. There is an obvious advantage in choosing generators of small weight as it simplifies the corresponding quantum measurements. Even though with fault-tolerant measurement protocols one could measure operators involving many qubits (e.g., in the case of concatenated codes[2]), it is much easier to measure operators which involve only a few qubits. Thus, we expect any large quantum code of any use to be degenerate. The ultimate case of degeneracy are w -limited quantum LDPC codes, where every stabilizer generator involves no more than w qubits.

Existence of a finite error correction threshold requires an infinite code family with divergent distances. For example, in codes with a finite *relative distance* $\delta \equiv d/n$ at large n , uncorrelated single-qubit errors occurring with probabilities $p < \delta/2$ can be corrected with certainty. The subject of this work are codes with *sublinear distance* scaling[20–23], e.g., power-law $d \propto n^\alpha$, with $\alpha < 1$. Here, at large n , any likely error will have weight pn which is much bigger than the distance.

We consider three simple error models[31]: quantum depolarizing channel, where with probability p an incoming qubit is replaced by a qubit in a random state, without notifying the observer; independent X/Z errors, where Pauli operators X and Z are applied to each qubit with probabilities p_X and p_Z , respectively, and the quantum erasure channel, where with probability y each qubit is replaced by an “erasure state” $|2\rangle$ orthogonal to both

$|0\rangle$ and $|1\rangle$. We will also consider FT using phenomenological error model where measurement errors happen independently with probability q . Such an error just results in the syndrome bit measured incorrectly; it does not affect the state of the qubits.

Below we consider infinite sequences of quantum codes whose distances scale with n at least logarithmically,

$$d \geq D \ln n, \quad D > 0. \quad (3)$$

Super-logarithmic scaling of the distance (including a power law $d \geq An^\alpha$ with $A, \alpha > 0$) gives $D \rightarrow \infty$. We summarize the constructed thresholds as follows:

Theorem 1. *Any sequence of quantum codes (3) with stabilizer generators of weights w or less can be decoded with a vanishing error probability if channel probabilities (y, p) for erasures and depolarizing errors satisfy $2(w - 1) \Upsilon(y, p) \leq e^{-1/D}$, where*

$$\Upsilon(y, p) \equiv y + (1 - y) \left\{ \frac{2p}{3} + 2 \left[\frac{p}{3} (1 - p) \right]^{1/2} \right\}. \quad (4)$$

Theorem 2. *Any sequence of CSS codes (3) with generator weights not exceeding w_X, w_Z can be decoded with vanishing error probabilities if channel probabilities (y, p_X, p_Z) for erasures and independent X/Z errors satisfy $(w_X - 1) \Upsilon_{\text{CSS}}(y, p_Z) \leq e^{-1/D}$, $(w_Z - 1) \Upsilon_{\text{CSS}}(y, p_X) \leq e^{-1/D}$, where*

$$\Upsilon_{\text{CSS}}(y, p) \equiv y + 2(1 - y) [p(1 - p)]^{1/2}. \quad (5)$$

FT case gives weaker versions of Theorems 1 and 2:

Theorem 3. *With the addition of phenomenological syndrome measurement errors with probability q , vanishing error rates are achieved if (a) error probabilities for stabilizer codes in Theorem 1 satisfy*

$$4[q(1 - q)]^{1/2} + 2wY(y, p) \leq e^{-1/D}, \quad (6)$$

(b) error probabilities for CSS codes in Theorem 2 satisfy

$$\begin{aligned} 4[q(1 - q)]^{1/2} + w_X Y_{\text{CSS}}(y, p_Z) &\geq e^{-1/D}, \\ 4[q(1 - q)]^{1/2} + w_Z Y_{\text{CSS}}(y, p_X) &\geq e^{-1/D}. \end{aligned} \quad (7)$$

Our analysis is based on counting irreducible undetectable operators:

Definition 1. *For a given stabilizer code \mathcal{Q} , an undetectable operator is called irreducible if it cannot be decomposed as a product of two undetectable Pauli operators with support on non-empty disjoint sets of qubits.*

This definition implies:

Lemma 4. *Any undetectable operator $E \in \mathcal{P}_n$ can be written as $E = \prod_i J_i$, where undetectable operators $J_i \in \mathcal{P}_n$, $\text{wgt } J_i \neq 0$, are irreducible and pairwise disjoint.*

For a given code, let $\mathcal{U} \subset \mathcal{P}_n \setminus \mathcal{S}$ denote the set of all non-trivial irreducible undetectable Pauli operators.

Given some error probability function $P(E)$, consider a syndrome-based decoder which returns the Pauli operator $E \in \mathcal{P}_n$ that produces the given syndrome and maximizes $P(E)$. Notice that this is not a true maximum-likelihood (ML) decoder since we are ignoring contributions of errors degenerate with E . Using an analogy with statistical mechanics[5, 32], ML decoding corresponds to minimizing the free energy; here we ignore entropy contribution resulting from degenerate errors and just minimize the *energy* $\varepsilon(E) \equiv -\ln P(E)$. Such a procedure is intrinsically sub-optimal; thus a lower bound for decoding threshold we get is also a lower bound for syndrome-based ML decoding.

Now, let $E \in \mathcal{P}_n$ be an error that actually happened, and E' be the same-syndrome Pauli operator which minimizes the energy $\varepsilon(E)$. This error can in principle be found, e.g., by an exhaustive search. The product $E'E^\dagger$ is undetectable, it satisfies Lemma 4, which gives a decomposition $E'E^\dagger = \prod_i J_i$ into irreducible undetectable operators, $J_i \in \mathcal{S} \cup \mathcal{U}$. Since the operators J_i are mutually disjoint, none of them can decrease the energy of E' , $\varepsilon(J_i E') \geq \varepsilon(E')$. Otherwise E' would not be the smallest-energy error with the same syndrome. Thus found minimal-energy error E' is correct iff $E'E^\dagger$ is trivial, which implies that all irreducible components must be members of the stabilizer, $J_j \in \mathcal{S}$ (up to a phase).

Otherwise, there is an irreducible operator $U \in \mathcal{U}$ which does not increase the energy of the original error E , $\varepsilon(UE) \leq \varepsilon(E)$. Let $\mathcal{B}(U) \equiv \{E \in \mathcal{P}_n : \varepsilon(UE) \leq \varepsilon(E)\}$ be the full set of such “bad” errors for a given $U \in \mathcal{U}$. Minimum-energy decoding gives vanishing error rate if

$$\text{Prob} \left[E : E \in \bigcup_{U \in \mathcal{U}} \mathcal{B}(U) \right] \rightarrow 0, \quad n \rightarrow \infty. \quad (8)$$

We bound the probability (8) by the sum of probabilities to encounter a “bad” error from each $\mathcal{B}(U)$; this gives the following sufficient condition for error-free decoding:

$$\sum_{U \in \mathcal{U}} \text{Prob} [E : E \in \mathcal{B}(U)] \rightarrow 0, \quad n \rightarrow \infty. \quad (9)$$

For uncorrelated errors only the qubits in the support of U affect the probabilities in Eq. (9). Furthermore, with uniform error distributions, these probabilities depend only on the weights $m \equiv \text{wgt } U$ of the operators $U \in \mathcal{U}$. For example, in the case of erasures with single-qubit probability y , a bad error must cover the entire support of U , which gives simply $\text{Prob}[E : E \in \mathcal{B}(U)] = y^{\text{wgt}(U)}$. Let N_m denote the number of operators $U \in \mathcal{U}$ of weight $m \equiv \text{wgt } U$. Since members of the stabilizer group are excluded from \mathcal{U} , $N_m = 0$ for $m < d$. Thus, in the case of the erasure channel, the condition (9) is equivalent to

$$\sum_{m \geq d} N_m y^m \rightarrow 0, \quad n \rightarrow \infty. \quad (10)$$

To construct an upper bound for N_m , we use a simplified version of the cluster-enumeration algorithm originally designed for finding the distance of a quantum

LDPC code[33, 34]. Let us assume that the r stabilizer generators G_i are ordered by weight, $\text{wgt } G_i \leq \text{wgt } G_{i+1}$, $1 \leq i < r$. Start by placing either of $\{X, Y, Z\}$ at a position $j \in \{0, \dots, n-1\}$ and place the corresponding Pauli operator as the only element of the list of the components of the operator being constructed. At every subsequent step, take the generator G_i corresponding to a non-zero syndrome bit with the smallest index i , and choose any position j in the support of G_i that is not yet selected; there are up to $\text{wgt } G_i - 1$ choices. Choose a single-qubit Pauli different from the term present at the position j in the expansion (1) of G_i , and add it to the list. This sets the syndrome bit s_i to zero without modifying any of the existing entries in the list. At every step of the recursion, zero syndrome means a completed undetectable cluster; no available positions to correct a chosen syndrome bit means recursion got stuck. In either case we need to go back one step by removing the element last added to the list. The procedure stops when we exhaust all choices.

If we limit the recursion to depth m , we are only going to construct operators of weight up to m . There are $3n$ possible choices for the first step, and up to $2(\text{wgt } G_i - 1)$ for each subsequent step. In the case of a w -limited quantum LDPC code, this means no more than

$$\bar{N}_m = 3n[2(w-1)]^{m-1} \quad (11)$$

recursion paths to construct operators of weight up to m . By construction, the algorithm returns only undetectable operators. While not all of them are irreducible, it is important that *all irreducible operators* of weight m are constructed with depth- m recursion. Indeed, for a given $U \in \mathcal{U}$, we just have to start with a non-trivial term in the corresponding expansion (1), and keep choosing only such terms at every step—the recursion will result in the list corresponding to U after exactly m steps. The procedure cannot end earlier since U is irreducible, and it cannot continue past the m -th step since U is undetectable. Notice also that we can select the positions from the support of U in any order as long as U is irreducible. This is in contrast to the case of an undetectable but reducible operator, see Fig. 1(b).

These arguments show that \bar{N}_m in Eq. (11) is an upper bound for the number N_m of the irreducible operators $U \in \mathcal{U}$ with weight $\text{wgt } U = m$, $\bar{N}_m \geq N_m$.

In the case of CSS codes, it is convenient to introduce the sets $\mathcal{U}_X \subset \mathcal{U}$ and $\mathcal{U}_Z \subset \mathcal{U}$ of non-trivial irreducible undetectable operators which are composed only of X and only of Z operators respectively, and denote $N_m^{(\mu)}$ the number of weight- m operators in \mathcal{U}_μ , $\mu \in \{X, Z\}$. For codes in Theorem 2, this gives improved bounds, e.g.,

$$N_m^{(X)} \leq \bar{N}_m^{(X)} \equiv n(w_X - 1)^{m-1}, \quad (12)$$

A bound for $N_m^{(Z)}$ can be obtained from Eq. (12) by exchanging the labels $X \leftrightarrow Z$.

We illustrate the cluster enumeration procedure on the toric code $[[2L^2, 2, L]]$, a CSS code with $w_X = w_Z = 4$ generators local in two dimensions. The qubits are placed

on the bonds of an $L \times L$ square lattice with periodic boundary conditions along both bond directions. The stabilizer generators are the plaquette and vertex operators, $A_{\square} = \prod_{j \in \square} X_j$ and $B_+ = \prod_{j \in +} Z_j$, see Fig. 1(a). A type- X cluster can be started by placing an X operator anywhere, which makes the two operators B_+ on the neighboring vertices unhappy (the corresponding syndrome bits are non-zero). Either can be corrected by placing an additional X operator on one of the remaining three open bonds adjoining the corresponding vertex. This produces an additional unhappy operator B_+ at the other end of the bond, etc. An undetectable cluster corresponds to a closed walk (cycle). Any cycle can be constructed this way. A topologically trivial cycle produces a member of the stabilizer group, while a cycle winding an odd number of times over one or both periodicity directions corresponds to a logical operator. Further, a self-avoiding closed walk corresponds to an irreducible undetectable operator, while a self-intersecting cycle produces an operator which can be decomposed into a product of two or more disjoint cycles, see Fig. 1(b).

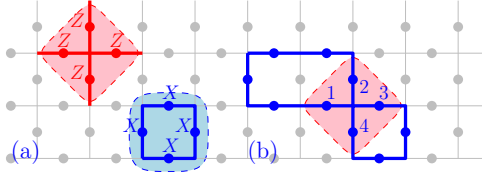


FIG. 1. (color online) Structure of the toric code. (a) Plaquette A_{\square} (shaded rounded square) and vertex B_+ (shaded diamonds) operators constructed as products of four Pauli Z and Pauli X operators respectively. (b) A reducible cluster (the corresponding operator can be split into a product of two undetectable operators on non-overlapping subsets) which will be counted as one or two clusters depending on the order in which the numbered qubits are chosen.

Combining Eq. (10) and the bound $N_m \leq \bar{N}_m$, see Eq. (11), we can prove a simplified version of Theorem 1 for erasure errors only. Namely, consider the sum

$$Q_d(y) \equiv \sum_{m \geq d} \bar{N}_m y^m = \frac{3ny[2y(w-1)]^{d-1}}{1-2y(w-1)}, \quad (13)$$

where we require $2y(w-1) < 1$ for absolute convergence. At large n , $Q_d(y)$ converges to zero as long as $n[2y(w-1)]^d \rightarrow 0$. This is true for any $y < e^{-1/D}/2(w-1)$ for codes in Eq. (3). The sum (10) is majored term by term by Eq. (13). This proves a version of Theorem 1 for the erasures only, and gives a lower bound for erasure threshold, $y_c \geq e^{-1/D}/2(w-1)$. In the case of distance scaling as a power-law or faster, the sum (13) asymptotically vanishes anywhere within the convergence radius, $y < [2(w-1)]^{-1}$, and we can just set $e^{-1/D} \rightarrow 1$.

The proof of Theorem 1 for the combined erasure and depolarizing errors in the case of generic w -limited quantum LDPC codes, or Theorem 2 with combined erasure and independent X/Z errors for CSS codes can be done

in a similar fashion if we notice that the corresponding probabilities in Eq. (9) can be bounded from above as in Eq. (10), with some effective erasure rate $\Upsilon \geq y$. The complete analysis is given in the Appendix A.

Our arguments so far apply in the conventional “code-capacity” setting which assumes that syndrome measurement is done ideally. In the case of quantum codes, more important is the fault-tolerant case where errors may occur at any time during syndrome measurement[2, 4, 35–39]. Such a complete analysis is beyond the scope of this work. Instead, we give a simplified estimate based on a phenomenological error model, which assumes that measured syndrome bits can have errors, but otherwise there is no effect on the qubits[5, 15, 25]. Error correction involves repeated syndrome measurement cycles and an auxiliary code which combines the syndromes measured in subsequent cycles. We only consider the simplest case where repetition code is used for combining the syndromes. For a CSS code, with equal uncorrelated qubit and syndrome errors $q = p_X = p_Z$, the net effect is equivalent to increasing the weights of stabilizer generators in Eq. (12) and in Theorem 2 by two, $w \rightarrow w+2$. With the surface codes, decoding corresponds to minimal-weight matching of chains in three dimensions[5]. For a more general result, we have to bound the number of weight- m clusters N_{m,m_q} which include m_q “qubit” Pauli operators, and $m-m_q$ binary syndrome errors. Statements of Theorem 3 follow from the bound $N_{m,m_q} \leq \bar{N}_{m,m_q}$,

$$\bar{N}_{m,m_q} \equiv (3n+r) \binom{m-1}{m_q} w^{m_q} 2^{m-m_q-1}. \quad (14)$$

This derivation of this expression and the details of the proof are given in the Appendix B.

How tight are the computed bounds? For the toric code ($w_X = w_Z = 4$), the erasure threshold is $y_c = 0.5$ and the ML threshold for independent X/Z errors is $p_{Zc} = p_{Xc} \approx 0.11$, compared with $y_c^* = 1/3$ and $p_{Zc}^* \approx 0.029$ expected from Theorem 2. We also checked the accuracy of Eq. (12) by enumerating irreducible clusters numerically (see Appendix D) and fitting with $\ln N_m = A + \zeta_w m$, where $\zeta_w \leq w-1$ for CSS codes with row weight w was expected from Eq. (12). In particular, we got $\zeta_6 \approx 4.76$, $\zeta_7 \approx 5.74$, $\zeta_8 \approx 5.79$ and $\zeta_9 \approx 6.78$, indicating that our bounds for N_m are relatively tight.

In conclusion, we constructed lower bounds on the thresholds of weight-limited quantum LDPC codes with sublinear distances scaling logarithmically or faster with the code length n . These bounds are based on estimating the number of logical operators which cannot be decomposed into a product of disjoint undetectable operators. The resulting analytical expressions combine probabilities of erasures, depolarizing errors (independent X/Z errors for CSS codes), and syndrome measurement errors using a phenomenological error model. These bounds are much stronger than those constructed previously[25], and they have a different dependence on the code parameters. In particular, we no longer require that each qubit be involved in a limited number of stabilizer generators. Qual-

itatively, the main difference is that the present analysis is no longer based on percolation theory.

This technique could be applicable not only for weight-limited LDPC codes, but also for more general degenerate codes, where the corresponding scaling of N_m can be calculated numerically or analytically (e.g., in the case of concatenated codes). It would be interesting to see if a finite FT threshold exists for finite-rate and finite-relative distance quantum LDPC codes constructed by Bravyi and Hastings[24]. Another potential application would be the analysis of fault-tolerance of subsystem codes, e.g., a subclass of those constructed in Ref. [40].

Our bounds can be also used to limit the parameters of quantum LDPC codes. In particular, combining our lower bound $y_c^{(\text{CSS})} \geq 1/(w-1)$ for erasure threshold from Theorem 2 with the trivial upper bound $y_c \leq (1-R)/2$ suggests that CSS LDPC codes with super-logarithmic distance scaling do not exist for $R > 1 - 2/(w-1)$. In the case of $w = 4$ codes this gives $R \leq 1/3$, whereas the only known example of such codes is $R = 0$ (toric codes). These can be further improved by using more accurate upper bounds constructed specifically for quantum LDPC codes in Ref. 27.

Also, as was pointed to us by Pastawski and Yoshida, our bounds on erasure thresholds can be combined with their upper bound[28] for codes which include non-trivial transversal logical gates from m -th level of the Clifford hierarchy[41], $y_m \leq 1/m$. Thus, e.g., only CSS codes with generators of weight $w \geq m+1$ may include such logical gates. Such codes are useful in constructing universal sets of FT gates acting on logical qubits directly, without the need for decoding; a set must include at least one non-Clifford operator ($m > 2$). We note that the analysis in Refs. 28 and 41 is largely based on the cleaning lemma[42, 43] which utilizes the notion of correctable subsets. These are complementary to our irreducible undetectable operators (Def. 1); it would be interesting to check if this relation could help extending the bounds constructed in Ref. 42 to general LDPC codes.

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Appendix A: Effective erasure probabilities

Here we derive Eqs. (5) and (4).

1. CSS code with erasures and independent X/Z errors

For a given CSS code, consider an X -type undetectable operator $U \in \mathcal{U}_X$ of weight m . Fix erasure probability y and X qubit error probability $p \equiv p_X$. We are not concerned with Z errors since these do not affect the Z -type stabilizer generators used to detect the error considered here. An erasure with known location supersedes regular qubit error. Thus, the probability of an error E with a erasures and b non-overlapping X -type errors in the subset of qubits of weight m is

$$P_E = \binom{m}{a} y^a (1-y)^{m-a} \binom{m-a}{b} p^b (1-p)^{m-a-b}. \quad (\text{A1})$$

The probability of an “inverted” error EU (notice that this does not affect erasures)

$$P_{EU} = \binom{m}{a} y^a (1-y)^{m-a} \binom{m-a}{b} (1-p)^b p^{m-a-b}.$$

The error E is “bad”, $E \in \mathcal{B}(U)$, if the inverted error has the same or larger probability; this gives:

$$(2b + a - m) \ln \frac{1-p}{p} \geq 0. \quad (\text{A2})$$

Summation of probabilities (A1) gives the net probability to encounter a “bad” error for $U \in \mathcal{U}$, wgt $U = m$:

$$\begin{aligned} P_m &= \sum_{2b+a-m \geq 0} P_E(a, b) \\ &= \sum_{2b+a-m \geq 0} \binom{m}{a} \binom{m-a}{b} y^a (1-y)^{m-a} [p(1-p)]^{(m-a)/2} \underbrace{\left(\frac{p}{1-p} \right)^{(2b+a-m)/2}}_{\leq 1} \end{aligned}$$

The marked term in the last line is smaller or equal than one in the summation region, the “bad” region $\mathcal{B}(U)$. We make an upper bound by dropping this term from the product and extending the summation to all values of $a \geq 0$, $b \geq 0$ such that $a + b \leq m$. The summation gives an exponent, with the base of the exponent being the effective

erasure probability in Eq. (5):

$$\begin{aligned}
P_m &\leq \sum_{a,b} \binom{m}{a} \binom{m-a}{b} y^a (1-y)^{m-a} [p(1-p)]^{(m-a)/2} \\
&= \sum_{a=0}^m \binom{m}{a} y^a \left\{ 2(1-y)[p(1-p)]^{1/2} \right\}^{m-a} \\
&= \left\{ y + 2(1-y)[p(1-p)]^{1/2} \right\}^m \\
&\equiv [\Upsilon_{\text{CSS}}(y, p)]^m.
\end{aligned}$$

2. Generic stabilizer code with erasures and depolarizing errors

For a given stabilizer code, consider an undetectable operator $U \in \mathcal{U}$ of weight m . Fix erasure probability y and depolarizing error probability p . Split the total error weight into a erasures and $b = b' + b''$ depolarizing errors, with the single-qubit Pauli errors in b' positions matching those in U , and errors in the remaining b'' positions different from the operators in U . Probability of such an error is

$$P_E(a, b', b'') = \binom{m}{a} \binom{m-a}{b} \binom{b}{b'} y^a (1-y)^{m-a} \left(\frac{p}{3}\right)^{b'} \left(2\frac{p}{3}\right)^{b''} (1-p)^{m-a-b'-b''}.$$

The probability of the corresponding “inverted” error EU (notice that the contribution of the a erasures or b'' differing positions remain unaffected):

$$P_{EU}(a, b', b'') = \binom{m}{a} \binom{m-a}{b} \binom{b}{b'} y^a (1-y)^{m-a} \left(\frac{p}{3}\right)^{m-a-b'-b''} \left(2\frac{p}{3}\right)^{b''} (1-p)^{b'}.$$

Respectively, in a bad region $\mathcal{B}(U)$: $P_{EU} \geq P_E$, we have $2b' + b'' + a - m \geq 0$. Using the same trick as before, we have an upper bound for the total probability of a bad error in a cluster of size m :

$$\begin{aligned}
P_m &= \sum_{E \in \mathcal{B}(U)} P_E \quad \left(\text{denote } x \equiv \frac{p/3}{1-p} \right) \\
&= \sum_{E \in \mathcal{B}(U)} \binom{m}{a} \binom{m-a}{b} \binom{b}{b''} y^a (1-y)^{m-a} \left(2\frac{p}{3}\right)^{b''} x^{(2b'+b''+a-m)/2} \left[\left(\frac{p}{3}\right)(1-p)\right]^{(m-a-b'')/2} \\
&\leq \sum_{a,b,b''} \binom{m}{a} \binom{m-a}{b} \binom{b}{b''} y^a (1-y)^{m-a} \left(2\frac{p}{3}\right)^{b''} \left[\left(\frac{p}{3}\right)(1-p)\right]^{(m-a-b'')/2} \\
&= \sum_{a,b} \binom{m}{a} \binom{m-a}{b} y^a (1-y)^{m-a} \left[\left(\frac{p}{3}\right)(1-p)\right]^{(m-a-b)/2} \left\{ 2\frac{p}{3} + \left[\left(\frac{p}{3}\right)(1-p)\right]^{1/2} \right\}^b \\
&= \sum_{a,b,b'} \binom{m}{a} y^a (1-y)^{m-a} \left\{ 2\frac{p}{3} + 2 \left[\left(\frac{p}{3}\right)(1-p)\right]^{1/2} \right\}^{m-a} \\
&= \left(y + (1-y) \left\{ 2\frac{p}{3} + 2 \left[\left(\frac{p}{3}\right)(1-p)\right]^{1/2} \right\} \right)^m.
\end{aligned}$$

The base of the exponent is the effective erasure probability $\Upsilon(y, p)$, see Eq. (4).

Appendix B: Outline the aux code construction for phenomenological error model

In this section, we use a binary representation of the Pauli operators[11, 19]. A Pauli operator can be mapped, up to a phase, to two binary strings, $\mathbf{v}, \mathbf{u} \in \{0, 1\}^n$,

$$U \equiv i^{m'} X^{\mathbf{v}} Z^{\mathbf{u}} \rightarrow (\mathbf{v}, \mathbf{u}), \quad (\text{B1})$$

where $X^{\mathbf{v}} = X_1^{v_1} X_2^{v_2} \dots X_n^{v_n}$ and $Z^{\mathbf{u}} = Z_1^{u_1} Z_2^{u_2} \dots Z_n^{u_n}$. A product of two quantum operators corresponds to a sum (mod 2) of the corresponding pairs $(\mathbf{v}_i, \mathbf{u}_i)$. Mapping each generator G_j , $j = 1, \dots, r$ of the stabilizer according

to Eq. (B1) gives rows of the binary generator matrix $G = (A_X|A_Z)$, with rows of A_X formed by \mathbf{v} and rows of A_Z formed by \mathbf{u} vectors. For generality, we also assume that the matrix G may also contain unimportant linearly dependent rows which are added after the mapping has been done (this corresponds to adding arbitrary products of stabilizer generators to the generators of \mathcal{S}). The commutativity of stabilizer generators corresponds to the following condition on the binary matrices A_X and A_Z :

$$A_X A_Z^T + A_Z A_X^T = 0 \pmod{2}. \quad (\text{B2})$$

In the case of the CSS codes, the generator matrix is block-diagonal,

$$G = \left(\begin{array}{c|c} G_X & 0 \\ 0 & G_Z \end{array} \right), \quad (\text{B3})$$

and the commutativity condition is just $G_X G_Z^T = 0$.

It is convenient to introduce the binary check matrix $H \equiv (A_Z|A_X)$ with the two blocks interchanged. Then, the commutativity condition (B2) becomes simply $HG^T = 0$. Similarly, an error operator in the form (B1) can be written as a binary vector $\mathbf{e}^T = (\mathbf{v}, \mathbf{u})$; the corresponding syndrome is just $\mathbf{s} = H\mathbf{e}$.

In the case of repeated syndrome measurements, in the phenomenological model, qubit errors accumulate while syndrome errors do not. Thus, if we denote qubit errors between $(i-1)$ th and i th measurement rounds as \mathbf{e}_i , and syndrome errors in these rounds as $\boldsymbol{\epsilon}_i$, $i = 1, \dots, m$, we have the equations

$$\begin{aligned} H\mathbf{e}_1 &= \mathbf{s}_1 + \boldsymbol{\epsilon}_1, \\ H(\mathbf{e}_1 + \mathbf{e}_2) &= \mathbf{s}_2 + \boldsymbol{\epsilon}_2, \\ &\dots \\ H(\mathbf{e}_1 + \mathbf{e}_2 + \dots + \mathbf{e}_m) &= \mathbf{s}_m + \boldsymbol{\epsilon}_m. \end{aligned}$$

Adding pairs of neighboring equations, and moving syndrome errors to the left, we obtain the following equations

$$\begin{aligned} H\mathbf{e}_1 + \boldsymbol{\epsilon}_1 &= \mathbf{s}_1, \\ H\mathbf{e}_2 + \boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 &= \mathbf{s}_1 + \mathbf{s}_2, \\ &\dots \\ H\mathbf{e}_m + \boldsymbol{\epsilon}_{m-1} + \boldsymbol{\epsilon}_m &= \mathbf{s}_{m-1} + \mathbf{s}_m. \end{aligned}$$

We notice that thus constructed combined code is not particularly good if treated as a classical binary code. Indeed, a low-weight error which consists of a single qubit error \mathbf{e}_s , $\text{wgt } \mathbf{e}_s = 1$, the syndrome error $\boldsymbol{\epsilon}_s = H\mathbf{e}_s$, and (in the case $s < m$) an identical single-qubit error $\mathbf{e}_{s+1} = \mathbf{e}_s$, will not be detected. On the other hand, for $s < m$, such an error obviously produces no mistake; we do not need to correct these errors just as we do not need to correct trivial degeneracy errors. In the case of $s = m$, the error cannot be detected at this cycle of measurements; it is convenient to reassign all such errors to the subsequent cycle (unless it corrects itself, such an error will be detected in the next round of measurements). This prescription is equivalent to setting $\boldsymbol{\epsilon}_m = \mathbf{0}$.

If we combine the (shortened) error and the syndrome vectors into unified vectors \mathbf{e} and \mathbf{s} , respectively, we can write these equations simply as $P\mathbf{e} = \mathbf{s}$, where

$$P = (I_m \otimes H_{r \times n}, R_{m \times (m-1)} \otimes I_r) \quad (\text{B4})$$

is a matrix formed by two blocks, with “ \otimes ” denoting Kronecker product, I_m an $m \times m$ identity matrix, and

$$R_{m \times (m-1)} \equiv \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & 1 & 1 & \\ & & & 1 \end{pmatrix} \quad (\text{B5})$$

is a transposed check matrix for the length- m repetition code. Further, it is easy to check that any combination of trivial degeneracy errors and undetectable self-corrected errors can be expressed as a linear combination of the rows of the matrix

$$Q = \begin{pmatrix} [R^T]_{(m-1) \times m} \otimes I_n & I_{m-1} \otimes [H^T]_{n \times r} \\ I_m \otimes G_{r' \times n}, & 0 \end{pmatrix}. \quad (\text{B6})$$

Obviously, $PQ^T = 0 \bmod 2$. In fact, the matrices P and Q can be viewed as CSS generators of a code similar to a hypergraph-product code[20, 21], with the difference that one of the constituent codes is a quantum, not a classical code. The parameters of such a code can be easily expressed in terms of those of the length- m repetition code and the original quantum code with the check matrix H and the generator matrix G . Namely, this code of length $N = mn + (m-1)r$ encodes exactly $K = k$ qubits with the distance $D = \min(d, m)$.

Such an auxiliary code results in a generalization of the three-dimensional line-matching for the case of the surface codes[5]; a similar construction has also been discussed in Refs. 15 and 25. For our purposes, it is important that for an original w -limited LDPC code, the check matrix P has row weight limited by $w + 2$, with up to w positions in the block corresponding to qubit errors, and the remaining one or two positions in the block corresponding to syndrome measurement errors.

Appendix C: Effective erasure probabilities with syndrome errors

Here, we derive Eqs. (6) and (7). The derivation is similar to that in Appendix A, with the difference that qubit errors and syndrome measurement errors have to be treated differently. Therefore, we consider an error E of the total weight m , with m_q positions in the “qubit” part corresponding to the first block of Eq. (B4), and the remaining $m - m_q$ positions in the “syndrome” part. To simplify the derivation, we omit the erasures.

1. CSS code with independent X/Z errors: FT case

Consider an X -type binary error e which produces zero syndrome with the check matrix (B4) where we only include the generator G_Z , see Eq. (B3). The error probabilities for qubits and syndrome bits are $p \equiv p_X$ and q , respectively. Then, the probability for e to cover b out of m_q qubit positions and f out of $m - m_q$ syndrome positions is

$$P_e = \binom{m_q}{b} \binom{m - m_q}{f} p^b (1-p)^{m_q-b} q^f (1-q)^{m-m_q-f},$$

whereas probability of the same error plus the codeword (error inverted) is

$$P_{e+c} = \binom{m_q}{b} \binom{m - m_q}{f} (1-p)^b p^{m_q-b} (1-q)^f q^{m-m_q-f}.$$

For a “bad” error, the ratio $P_{e+c}/P_e \geq 1$, which defines the bad-error region \mathcal{B} :

$$(2b - m_q) \ln \frac{1-p}{p} + (2f + m_q - m) \ln \frac{1-q}{q} \geq 0.$$

In the absence of syndrome measurement errors this goes over to $2b \geq m_q$, cf. Eq. (A2) with $a \rightarrow 0$. Now, we need to find the total probability of an error in \mathcal{B} :

$$\begin{aligned} P_{\text{bad}} &= \sum_{(b,f) \in \mathcal{B}} P_e(b, f) \\ &= \sum_{(b,f) \in \mathcal{B}} \binom{m_q}{b} \binom{m - m_q}{f} p^b (1-p)^{m_q-b} q^f (1-q)^{m-m_q-f} \\ &= \sum_{(b,f) \in \mathcal{B}} \binom{m_q}{b} \binom{m - m_q}{f} \underbrace{\left(\frac{p}{1-p} \right)^{(2b-m_q)/2} \left(\frac{q}{1-q} \right)^{(2f+m_q-m)/2}}_{\leq 1} [p(1-p)]^{m_q/2} [q(1-q)]^{(m-m_q)/2}. \end{aligned}$$

The marked portion of the expression does not exceed one in \mathcal{B} ; dropping it and extending the summation to all $b \leq m_q$, $f \leq m - m_q$, we obtain

$$P_{\text{bad}}(m, m_q) \leq 2^m [p(1-p)]^{m_q/2} [q(1-q)]^{(m-m_q)/2}. \quad (\text{C1})$$

We consider specifically the generators coming from the matrix in the form (B4), with weight up to $w \equiv w_Z$ in the qubit positions, and weight up to 2 in the syndrome positions. Then, the number of clusters can be bounded by

$$N_{m, m_q} \leq \overline{N}_{m, m_q}^{(\text{CSS})} = n \binom{m-1}{m_q} w^{m_q} 2^{m-m_q-1} \leq n \binom{m}{m_q} w^{m_q} 2^{m-m_q},$$

cf. Eq. (14). This is slightly worse than the bound (12) since we do not know whether the particular non-zero check bit originated from a qubit or a syndrome error. Overall, the net probability for “bad” error of combined weight exceeding the distance d can be bounded by

$$\begin{aligned}
 P_{\text{bad}}^{(\text{tot})} &\leq \sum_{m \geq d} \overline{N}_{m, m_q}^{(\text{CSS})} P_{\text{bad}}(m, m_q) \\
 &\leq n \sum_{m \geq d} \binom{m}{m_q} 2^m [w^2 p(1-p)]^{m_q/2} [4q(1-q)]^{(m-m_q)/2} \\
 &= n \sum_{m \geq d} \left(4[q(1-q)]^{1/2} + 2w[p(1-p)]^{1/2} \right)^m.
 \end{aligned} \tag{C2}$$

Analysis of the convergence for codes (3) gives the sufficient condition

$$4[q(1-q)]^{1/2} + 2w[p(1-p)]^{1/2} \leq e^{-1/D}, \tag{C3}$$

a special case of Eqs. (7) for $y \rightarrow 0$.

The derivation of the complete expressions (7), and of Eq. (6) for the case of combined erasures, depolarizing errors, and syndrome errors is similar.

Appendix D: Numerics

We implemented the cluster-enumeration algorithm presented in the main text numerically on Mathematica. Fig. 2 shows N_m computed for the hyperbicycle code[23] $[[168, 6, 12]]$, a CSS code with generators of weight $w_X = w_Z = 6$ produced from the cyclic binary code $[7, 3, 4]$; a hypergraph-product code[20] $[[1508, 100, 6]]$ with row weights limited by $w_X = w_Z = 7$ produced from a Gallager code $[32, 10, 6]$ with row weights 4; and two matching three-dimensional codes $[[2940, 6, 12]]$ and $[[12568, 100, 6]]$ with the binary check matrix (B4). The latter codes used $m = 12$ and $m = 6$ layers respectively, and have the respective row weights bounded by $w = 8$ and $w = 9$.

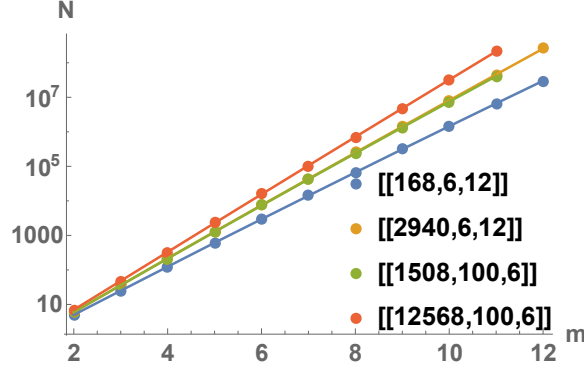


FIG. 2. (Color online) Numbers of undetectable clusters computed numerically for several codes as indicated. See text for the details of the codes. The fits $\ln N = A + \zeta m$ give slopes $\zeta = 4.76261, 5.7921, 5.74431, 6.7889$, respectively.